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1996 J. Phys. A: Math. Gen. 29 3075

(http://iopscience.iop.org/0305-4470/29/12/015)

J. Phys. A: Math. Gen. **29** (1996) 3075–3088. Printed in the UK

On nonlinear angular momentum theories, their representations and associated Hopf structures

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Received 15 January 1996

Abstract. Nonlinear $sl(2)$ algebras subtending generalized angular momentum theories are studied in terms of undeformed generators and bases. We construct their unitary irreducible representations in such a general context. The linear $sl(2)$ case as well as its q-deformation are easily recovered as specific examples. Two other physically interesting applications corresponding to the so-called Higgs and quadratic algebras are also considered. We show that these two nonlinear algebras can be equipped with a Hopf structure.

1. Introduction

Quantum groups [1] evidently appear as algebras with an infinite set of products of generators on the right-hand side of their commutation relations. If we limit the order of such products, we also get particular generalizations of ordinary Lie algebras that we simply refer to here as nonlinear algebras defined in following section: let us mention in particular that $\mathcal{W}$-finite algebras [2] belong to that category but also that there are known examples like the Higgs algebra [3] (containing *cubic* terms) and like the so-called *quadratically* nonlinear algebras [4]. Such specific nonlinear algebras have recently been investigated by Roček [5] and related by Quesne [6] to generalized deformed parafermions [7] which can be exploited in the study of the spectra of Morse and modified Pöschl–Teller Hamiltonians [8] as well as of parasupersymmetric Hamiltonians [9].

We are interested in some generalizations of the (so important) angular momentum theory being subtended by the real forms of the complex Lie (Cartan) algebra A_1 [10] to nonlinear extensions of A_1. In particular, we plan to study the representations associated with such nonlinear algebras. This is the first aim of our study. The second is connected with the possibility, where feasible, of endowing these nonlinear algebras with a Hopf structure [1]. Consequently, the paper is organized as follows.

In section 2, we study a specific series (admitting only odd powers) of *nonlinear* $sl(2)$ algebras subtending generalized angular momentum theories and construct their unitary irreducible representations. In section 3, we give a generalization of nonlinear algebras

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when the starting point was $\mathcal{U}_q(sl(2))$. In section 4, we show that the linear $sl(2)$ case as well as its q-deformation [1] are particular examples of our developments. Some comments about the Hopf structure of these nonlinear algebras are given in section 5. The specific *cubic* context and some comments about the Hopf structure are then considered in section 6. We also show that there exist other new families of representations when a specific choice of the diagonal generator is considered. Then, we study the *quadratic* context in section 7 by exploiting the above choice although this nonlinear algebra does not belong to the specific series. Some comments about this quadratic $sl(2)$ algebra are also given. Finally, section 8 is devoted to general comments and conclusions in connection with other recent proposals.

2. Representation theory of nonlinear $sl(2)$ algebras

In terms of the ladder generators $J_\pm$ and the diagonal one J_3, the very well known linear $sl(2)$ algebra is characterized by the commutation relations [11]

$$[J_+, J_-] = 2J_3 \tag{2.1}$$

$$[J_3, J_\pm] = \pm J_\pm \tag{2.2}$$

and by the Casimir operator

$$\mathcal{C} = \tfrac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \tag{2.3}$$

acting on an orthogonal basis denoted as usual by $\{|j, m\rangle\}$. In fact, we have the well known results

$$\mathcal{C}|j, m\rangle = j(j+1)|j, m\rangle \qquad j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2}, \ldots \tag{2.4}$$

$$J_3|j, m\rangle = m|j, m\rangle \qquad m = -j,\ -j+1, \ldots, j-1,\ j \tag{2.5}$$

$$J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle \tag{2.6}$$

which characterize all the unitary irreducible representations of this simple Lie algebra.

Let us consider the algebras that we decide to call nonlinear $sl(2)$ algebras due to the nonlinear terms appearing on the right-hand sides of the following commutation relations (in correspondence with the ones given by equations (2.1) and (2.2)), i.e.

$$[\hat{J}_+, \hat{J}_-] = \sum_{p=0}^{N} \beta_p (2\ \hat{J}_3)^{2p+1} \tag{2.7}$$

$$[\hat{J}_3, \hat{J}_\pm] = \pm \hat{J}_\pm \tag{2.8}$$

where the hat indices help us to distinguish these modified structures with respect to the algebra $sl(2)$. In fact, let us define a new basis of the algebra subtended by $J_\pm$ and J_3 as follows:

$$\hat{J}_+ = J_+\ f^+(\mathcal{C}, J_3) \qquad \hat{J}_- = f^-(\mathcal{C}, J_3)\ J_- \tag{2.9}$$

and

$$\hat{J}_3 = J_3 \tag{2.10}$$

so that we evidently ensure the relations (2.7) and (2.8) for arbitrary functions f^+ and f^- in terms of the commuting operators $\mathcal{C}$ and J_3 if we require that, on the state $|j, m\rangle$, we have

$$(j+m)(j-m+1)f^+(j,m-1)f^-(j,m-1)$$
$$-(j-m)(j+m+1)f^+(j,m)f^-(j,m) = \sum_{p=0}^{N}\beta_p(2m)^{2p+1}. \qquad (2.11)$$

If $f^\pm$ are real functions of $\mathcal{C}$ and J_3, then hermiticity implies $f^+ = f^-$.

Let us point out that our choice (2.9) is such that the ladder generators can be seen as Hermitian conjugate ones and that equation (2.10) leaves the diagonal operator J_3 unchanged. Relatively fastidious calculations starting with equation (2.11) lead to the result

$$(j-m)(j+m+1)f^+(j,m)f^-(j,m) = \sum_{p=0}^{N}\beta_p 2^{2p+1}\left(\sum_{r=1}^{j} r^{2p+1} - \sum_{r=1}^{m} r^{2p+1}\right)$$
$$= \sum_{p=0}^{N}\beta_p 2^{2p+1}\left(\frac{1}{2p+2}j^{2p+2} + \frac{1}{2}j^{2p+1} + \frac{1}{2}\binom{2p+1}{1}B_1 j^{2p}\right.$$
$$-\frac{1}{4}\binom{2p+1}{3}B_2 j^{2p-2} + \frac{1}{6}\binom{2p+1}{5}B_3 j^{2p-4} - \cdots$$
$$-(-1)^p\frac{1}{2p}\binom{2p+1}{2p-1}B_p j^2 - \frac{1}{2p+2}m^{2p+2} - \frac{1}{2}m^{2p+1} - \frac{1}{2}\binom{2p+1}{1}B_1 m^{2p}$$
$$+\frac{1}{4}\binom{2p+1}{3}B_2 m^{2p-2} - \frac{1}{6}\binom{2p+1}{5}B_3 m^{2p-4} + \cdots$$
$$\left.+(-1)^p\frac{1}{2p}\binom{2p+1}{2p-1}B_p m^2)\right) \qquad (2.12)$$

where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}, \ldots,$ are Bernoulli numbers [12] appearing in this particular summation of series [12]. By dividing both sides of the above equality by $(j-m)(j+m+1)$, the final result can be put in the form

$$f^+(j,m)f^-(j,m) = \beta_0 + \sum_{k=1}^{N}\beta_k\frac{2^{2k}}{k+1}\left(\sum_{r=1}^{k}\sum_{s=0}^{r}(j(j+1))^s(m(m+1))^{r-s}\epsilon_r(k)\right) \qquad (2.13)$$

or, in terms of generators,

$$f^+(\mathcal{C},J_3)f^-(\mathcal{C},J_3) = \beta_0 + \sum_{k=1}^{N}\beta_k\frac{2^{2k}}{k+1}\left(\sum_{r=1}^{k}\epsilon_r(k)\sum_{s=0}^{r}\mathcal{C}^s(J_3(J_3+1))^{r-s}\right). \qquad (2.14)$$

In equations (2.13) and (2.14), we have introduced specific functions of k defined by the following relations:

$$\epsilon_k(k) = 1 \qquad (2.15)$$

and, for $j = 1, 2, \ldots, k-1$,

$$(-1)^{j+1}\left(\frac{k+1}{j}\right)\binom{2k+1}{2j-1}B_j$$
$$= \binom{k+1}{2j} + \epsilon_{k-1}(k)\binom{k}{2j-2} + \epsilon_{k-2}(k)\binom{k-1}{2j-4} + \cdots + \epsilon_{k-j}(k). \qquad (2.16)$$

In addition, let us also point out that we could rewrite equation (2.14) in the following form:

$$f^+(\mathcal{C}, J_3)f^-(\mathcal{C}, J_3) = \sum_{k=1}^{N+1}\alpha_k\left(\sum_{n=0}^{k-1}\mathcal{C}^{k-1-n}(J_3(J_3+1))^n\right) \tag{2.17}$$

leading to simple identifications between the α- and β-coefficients. In fact, we have

$$\alpha_1 = \beta_0 \qquad \alpha_l = \sum_{k=l-1}^{N}\beta_k\frac{2^{2k}}{k+1}\epsilon_{l-1}(k) \qquad l = 2,\ 3,\ldots,\ N+1. \tag{2.18}$$

With this last set of information, the corresponding representations are simpler. Indeed, we get

$$\hat{J}_\pm|j,m\rangle = \left(\sum_{k=1}^{N+1}\alpha_k\big((j(j+1))^k - (m(m+1))^k\big)\right)^{1/2}|j,m\pm 1\rangle \tag{2.19}$$

and the commutation relation (2.7) becomes

$$[\hat{J}_+, \hat{J}_-] = 2\sum_{n=1}^{N+1}\alpha_n\sum_{r=0}^{R_n}\binom{n}{2r+1}\hat{J}_3^{2n-2r-1} \tag{2.20}$$

where $R_n = \frac{1}{2}(n-2)$ for even n and $R_n = \frac{1}{2}(n-1)$ for odd n. We thus relate the α- and β-coefficients in the other way (with respect to equations (2.18)) by

$$\beta_p = 2^{-2p}\sum_{k=p+1}^{2p+1}\alpha_k\binom{k}{2k-2p-1} \qquad p = 0,\ 1,\ldots,\ N. \tag{2.21}$$

Up to these choices, we have obtained at this stage some new information on irreducible representations of the nonlinear $sl(2)$ algebra for arbitrary N. We have to add more specific arguments in order to get all the representations as it will appear in what follows.

Now, let us give the explicit expressions of the deformed generators $\hat{J}_\pm$. According to equations (2.9) and (2.10), we have

$$\hat{J}_+ = J_+\left(\sum_{k=1}^{N+1}\sum_{r=0}^{k-1}\alpha_k\mathcal{C}^{k-1-r}(J_3(J_3+1))^r\right)^{1/2} \tag{2.22}$$

and

$$\hat{J}_- = \left(\sum_{k=1}^{N+1}\sum_{r=0}^{k-1}\alpha_k\mathcal{C}^{k-1-r}(J_3(J_3+1))^r\right)^{1/2}J_- \tag{2.23}$$

or

$$\hat{J}_+ = J_+\left(\sum_{k=1}^{N+1}\alpha_k\frac{\mathcal{C}^k - (J_3(J_3+1))^k}{\mathcal{C} - J_3(J_3+1)}\right)^{1/2} \tag{2.24}$$

and

$$\hat{J}_- = \left(\sum_{k=1}^{N+1}\alpha_k\frac{\mathcal{C}^k - (J_3(J_3+1))^k}{\mathcal{C} - J_3(J_3+1)}\right)^{1/2}J_-. \tag{2.25}$$

Moreover, if we define

$$\phi(x) = \sum_{k=1}^{N+1}\alpha_k\ x^k \tag{2.26}$$

these generators become

$$\hat{J}_+ = J_+ \left(\frac{\phi(\mathcal{C}) - \phi(J_3(J_3+1))}{\mathcal{C} - J_3(J_3+1)} \right)^{1/2} \tag{2.27}$$

$$\hat{J}_- = \left(\frac{\phi(\mathcal{C}) - \phi(J_3(J_3+1))}{\mathcal{C} - J_3(J_3+1)} \right)^{1/2} J_- \tag{2.28}$$

and the corresponding Casimir operator is

$$\hat{\mathcal{C}} = \tfrac{1}{2} \left(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ + \phi(\hat{J}_3(\hat{J}_3+1)) + \phi(\hat{J}_3(\hat{J}_3-1)) \right) = \phi(\mathcal{C}). \tag{2.29}$$

Remark. We can can also write (2.27) and (2.28) as (see the $\mathcal{U}_q(sl(2))$ case)

$$\hat{J}_+ = J_+ \left(\frac{\left(\psi \left(\sqrt{\mathcal{C} + \left(\frac{1}{2}\right)^2} \right) \right)^2 - \left(\psi(J_3 + \frac{1}{2}) \right)^2}{\left(\sqrt{\mathcal{C} + \left(\frac{1}{2}\right)^2} \right)^2 - \left(J_3 + \frac{1}{2} \right)^2} \right)^{1/2} \tag{2.30}$$

$$\hat{J}_- = \left(\frac{\left(\psi \left(\sqrt{\mathcal{C} + \left(\frac{1}{2}\right)^2} \right) \right)^2 - \left(\psi(J_3 + \frac{1}{2}) \right)^2}{\left(\sqrt{\mathcal{C} + \left(\frac{1}{2}\right)^2} \right)^2 - \left(J_3 + \frac{1}{2} \right)^2} \right)^{1/2} J_- \tag{2.31}$$

where

$$\phi(x) = \psi^2 \left(\sqrt{x + \tfrac{1}{4}} \right) - \psi^2 \left(\tfrac{1}{2} \right) \qquad \text{if} \quad \phi(0) = 0. \tag{2.32}$$

The relation between the deformed Casimir $\hat{\mathcal{C}}$ and $\mathcal{C}$ is given by

$$\sqrt{\hat{\mathcal{C}} + \left(\psi \left(\tfrac{1}{2} \right) \right)^2} = \psi \left(\sqrt{\mathcal{C} + \left(\tfrac{1}{2} \right)^2} \right). \tag{2.33}$$

Now, if ϕ is bijective, we evidently have

$$\mathcal{C} = \phi^{-1}(\hat{\mathcal{C}}) \tag{2.34}$$

and

$$J_+ = \hat{J}_+ \left(\frac{\phi^{-1}(\hat{\mathcal{C}}) - \hat{J}_3(\hat{J}_3+1)}{\hat{\mathcal{C}} - \phi(\hat{J}_3(\hat{J}_3+1))} \right)^{1/2} \tag{2.35}$$

$$J_- = \left(\frac{\phi^{-1}(\hat{\mathcal{C}}) - \hat{J}_3(\hat{J}_3+1)}{\hat{\mathcal{C}} - \phi(\hat{J}_3(\hat{J}_3+1))} \right)^{1/2} \hat{J}_-. \tag{2.36}$$

From this point of view bijective ϕ's are of particular interest. A similar discussion is valid for the function ψ.

3. A generalization

For equations (2.27) and (2.28), the starting point is $sl(2)$. One can take $\mathcal{U}_q(sl(2))$ (which is itself a nonlinear generalization of $sl(2)$) as the starting point and generalize that again by postulating

$$\hat{J}_\pm|j,m\rangle = \left(\sum_{k=1}^{N+1}\alpha_k\big(([j][j+1])^k - ([m][m+1])^k\big)\right)^{1/2}|j,m\pm 1\rangle \tag{3.1}$$

i.e.

$$\hat{J}_+ = J_+\left(\frac{\phi(\mathcal{C})-\phi([J_3][J_3+1])}{\mathcal{C}-[J_3][J_3+1]}\right)^{1/2} \tag{3.2}$$

$$\hat{J}_- = \left(\frac{\phi(\mathcal{C})-\phi([J_3][J_3+1])}{\mathcal{C}-[J_3][J_3+1]}\right)^{1/2}J_- \tag{3.3}$$

where

$$\mathcal{C} = \tfrac{1}{2}(J_+J_- + J_-J_+) + [J_3]^2. \tag{3.4}$$

For example, if we choose

$$\phi(x) = x + \frac{\beta}{[2]}x^2 \tag{3.5}$$

we obtain the following commutation relation:

$$[\hat{J}_+,\ \hat{J}_-] = [2J_3](1+\beta[J_3]^2). \tag{3.6}$$

For another choice, we can obtain

$$[\hat{J}_+,\ \hat{J}_-] = [\,[2J_3]_1]_2 \tag{3.7}$$

where,

$$[x]_i = \frac{q_i^x - q_i^{-x}}{q_i - q_i^{-1}} \qquad q_i \in \mathbb{C}. \tag{3.8}$$

One can ultimately even envisage a hierarchy of q-brackets generalizing the right-hand side of (3.7).

When one generalizes (2.19) as in (3.1), ($\hat{J}_\pm$, $q^{\pm\hat{J}_3}$) being expressed in terms of ($J_\pm$, $q^{\pm J_3}$) of $\mathcal{U}_q(sl(2))$, one can implement the standard Hopf structure of the latter (rather than starting from that of $sl(2)$) to construct $\hat{J}_\pm$ for product representations. Evidently the formalism of this section contains the results of the preceding one as limiting cases ($q \longrightarrow 1$).

4. The $sl(2)$ and $\mathcal{U}_q(sl(2))$ contexts

The nonlinear $sl(2)$ algebras given by equations (2.7) and (2.8) evidently contain the expected linear $sl(2)$ one as well as its q-deformation $\mathcal{U}_q(sl(2))$. The first one corresponds to $N = 0$, so that equations (2.7) with $\beta_0 = 1$ and (2.1) become identical while the second one is readily obtained by taking the limit $N \to \infty$ with the coefficients

$$\beta_p = \frac{2}{q-q^{-1}}\frac{(\log q)^{2p+1}}{(2p+1)!} \qquad p = 0,\ 1,\ldots \tag{4.1}$$

$$= \frac{1}{\sinh\delta}\frac{(\delta)^{2p+1}}{(2p+1)!} \qquad q = \exp\ \delta. \tag{4.2}$$

If, in the linear case, we evidently have

$$f^+(j,m)f^-(j,m) = 1 \qquad f^+ = f^- = 1 \tag{4.3}$$

ensuring that

$$\hat{J}_\pm = J_\pm \qquad \hat{J}_3 = J_3 \tag{4.4}$$

we point out in the q-deformation that [13]

$$f^+(j,m)f^-(j,m) = \frac{[j-m][j+m+1]}{(j-m)(j+m+1)} \tag{4.5}$$

where as usual

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{4.6}$$

By developing the right-hand side of (4.5), it is not difficult to show that it coincides with our expression (2.13) (for example) with the coefficients (4.1). This corresponds to the equality

$$\frac{1}{(j-m)(j+m+1)} \frac{\cosh\big(\delta(2j+1)\big) - \cosh\big(\delta(2m+1)\big)}{2\sinh^2\delta}$$

$$= \frac{\delta}{\sin\delta} + \sum_{k=1}^{\infty} \frac{2^{2k+1}\delta^{2k+1}}{(2k+2)!\ \sinh\delta} \sum_{r=1}^{k} \sum_{s=0}^{r} (j(j+1))^s (m(m+1))^{r-s} \epsilon_r(k) \tag{4.7}$$

where the corresponding functions $\epsilon_r(k)$ are given by (2.15) and (2.16).

Let us also mention that the quantum algebra $\mathcal{U}_q(sl(2))$ corresponds to the choice of the following bijective function introduced by (2.26):

$$\phi(J_3(J_3+1)) = [J_3][J_3+1] \qquad \phi(\mathcal{C}) = \left[\sqrt{\mathcal{C}+\tfrac{1}{4}} - \tfrac{1}{2}\right]\left[\sqrt{\mathcal{C}+\tfrac{1}{4}} + \tfrac{1}{2}\right] \tag{4.8}$$

with the bracket (4.6) so that in this context the generators (2.27) and (2.28) become

$$\hat{J}_+ = J_+ \left(\frac{\left[\sqrt{\mathcal{C}+\left(\frac{1}{2}\right)^2}\right]^2 - \left[J_3+\frac{1}{2}\right]^2}{\left(\sqrt{\mathcal{C}+\left(\frac{1}{2}\right)^2}\right)^2 - \left(J_3+\frac{1}{2}\right)^2} \right)^{1/2} \tag{4.9}$$

$$\hat{J}_- = \left(\frac{\left[\sqrt{\mathcal{C}+\left(\frac{1}{2}\right)^2}\right]^2 - \left[J_3+\frac{1}{2}\right]^2}{\left(\sqrt{\mathcal{C}+\left(\frac{1}{2}\right)^2}\right)^2 - \left(J_3+\frac{1}{2}\right)^2} \right)^{1/2} J_-. \tag{4.10}$$

The corresponding Casimir operator is then given by

$$\hat{\mathcal{C}} = \left[\sqrt{\mathcal{C}+\left(\tfrac{1}{2}\right)^2} - \tfrac{1}{2}\right]\left[\sqrt{\mathcal{C}+\left(\tfrac{1}{2}\right)^2} + \tfrac{1}{2}\right] \tag{4.11}$$

i.e.

$$\sqrt{\hat{\mathcal{C}} + \left[\tfrac{1}{2}\right]^2} = \left[\sqrt{\mathcal{C}+\left(\tfrac{1}{2}\right)^2}\right] \tag{4.12}$$

and, consequently,

$$\sqrt{\mathcal{C}+\left(\tfrac{1}{2}\right)^2}=\frac{1}{\delta}\ \text{arcsinh}\left(\sqrt{\hat{\mathcal{C}}+\left[\tfrac{1}{2}\right]^2}\sinh\delta\right). \tag{4.13}$$

These relations finally lead to

$$J_+=\hat{J}_+\left(\frac{\left(\frac{1}{\delta}\ \text{arcsinh}\left(\sqrt{\hat{\mathcal{C}}+\left[\frac{1}{2}\right]^2}\ \sinh\delta\right)\right)^2-(\hat{J}_3+\frac{1}{2})^2}{\left(\sqrt{\hat{\mathcal{C}}+\left[\frac{1}{2}\right]^2}\right)^2-[\hat{J}_3+\frac{1}{2}]^2}\right)^{1/2} \tag{4.14}$$

$$J_-=(J_+)^+=\left(\frac{\left(\frac{1}{\delta}\ \text{arcsinh}\left(\sqrt{\hat{\mathcal{C}}+\left[\frac{1}{2}\right]^2}\ \sinh\delta\right)\right)^2-(\hat{J}_3+\frac{1}{2})^2}{\left(\sqrt{\hat{\mathcal{C}}+\left[\frac{1}{2}\right]^2}\right)^2-[\hat{J}_3+\frac{1}{2}]^2}\right)^{1/2}\hat{J}_- \tag{4.15}$$

ensuring that we have the expected commutation relation

$$[\hat{J}_+,\hat{J}_-]=[2J_3]. \tag{4.16}$$

It has been shown by Curtright *et al* [13] that, from the well known $sl(2)$-cocommutative coproduct and (4.9)–(4.10), it is possible to characterize $\mathcal{U}_q(sl(2))$ by a cocommutative coproduct. Moreover, by the inverse map (4.14)–(4.15) and the non-cocommutative coproduct of $\mathcal{U}_q(sl(2))$ denoted by $\triangle_q$, we can also characterize the linear $sl(2)$ by a non-cocommutative one.

We do not go further into these directions due to our specific interest in *finite* values of $N \neq 0$ and more particularly in $N=1$, the first non-trivial value which has a direct connection with already studied physical contexts [14].

5. Hopf structure of nonlinear algebras

In what follows, we start by enlarging the term of enveloping algebra of $sl(2)$ to include square roots. Then, exploiting the well known fact [1] that the undeformed generators $J_\pm$ and J_3 admit a Hopf structure with the well known coproduct, counit and antipode given for example in the *cocommutative case* respectively by

(i) $$\triangle(J_\pm)=J_\pm\otimes 1+1\otimes J_\pm \tag{5.1}$$

$$\triangle(J_3)=J_3\otimes 1+1\otimes J_3 \tag{5.2}$$

leading to

$$\triangle(\mathcal{C})=\mathcal{C}\otimes 1+1\otimes\mathcal{C}+J_+\otimes J_-+J_-\otimes J_++2J_3\otimes J_3 \tag{5.3}$$

(ii) $$\varepsilon(J_\pm)=\varepsilon(J_3)=\varepsilon(\mathcal{C})=0 \tag{5.4}$$

(iii) $$S(J_\pm)=-J_\pm \qquad S(J_3)=-J_3 \qquad S(C)=C \tag{5.5}$$

we can deduce that our deformed generators $\hat{J}_\pm$ and $\hat{J}_3$ also satisfy the Hopf axioms, i.e. [1]:

$$(\text{id}\otimes\triangle)\triangle=(\triangle\otimes\text{id})\triangle \tag{5.6}$$

$$m(\text{id}\otimes S)\triangle=m(S\otimes\text{id})\triangle=i\circ\varepsilon \tag{5.7}$$

$$(\text{id}\otimes\varepsilon)\triangle=(\varepsilon\otimes\text{id})\triangle=\text{id}. \tag{5.8}$$

Now, the coproduct of our deformed generators is given by

$$\triangle(\hat{J}_+) = f^+\left(\triangle(\mathcal{C}), \triangle(J_3)\right) \triangle\left(J_+\right) \tag{5.9}$$

$$\triangle(\hat{J}_-) = \triangle(J_-) f^-\left(\triangle(\mathcal{C}), \triangle(J_3)\right) \tag{5.10}$$

it is not difficult to test that this coproduct is cocommutative, the same way of reasoning applying to the counit and antipode.

Let us remark that the right-hand sides of (5.9) and (5.10) are just an expansion of the generators $J_\pm$ and J_3. If the function ϕ (ψ) is bijective, equations (2.35)–(2.36), the Hopf structure *can be written* using *only* the *deformed* generators $\hat{J}_\pm$ and $\hat{J}_3$. If we take $\triangle_q$ given by Curtright et al [13] we can endow the nonlinear algebra (ϕ is bijective) by a non-cocommutative coproduct.

6. The cubic $sl(2)$ algebra

Let us now consider the $N = 1$-context leading, in (2.7), at most to the *cubic* power in the diagonal generator. This corresponds in particular to the nonlinear Higgs algebra [3], a symmetry one for the harmonic oscillator and the Kepler problems in a two-dimensional curved space. From (2.13), we immediately get

$$f^+(j,m) f^-(j,m) = \beta_0 + 2\beta_1\left(j(j+1) + m(m+1)\right) \tag{6.1}$$

leading to the Higgs algebra when $\beta_0 = 1$, $\beta_1 = \beta$. Then, we have the commutation relations

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_3 + 8\beta\hat{J}_3^3 \tag{6.2}$$

$$[\hat{J}_3, \hat{J}_\pm] = \pm\hat{J}_\pm. \tag{6.3}$$

By requiring that the ladder operators are Hermitian conjugate to each other, we have to fix

$$f^+(j,m) = \left(1 + 2\beta\left(j(j+1) + m(m+1)\right)\right)^{1/2} \tag{6.4}$$

so that the unitary irreducible representations of the Higgs algebra are given by

$$\hat{J}_\pm|j,m\rangle = \left((j\mp m)(j\pm m+1)\left(1+2\beta\left(j(j+1)+m(m\pm 1)\right)\right)\right)^{1/2}|j,m\pm 1\rangle \tag{6.5}$$

$$\hat{J}_3|j,m\rangle = m|j,m\rangle \tag{6.6}$$

where the parameter β is constrained by ensuring

$$1 + 2\beta\left(j(j+1) + m(m\pm 1)\right) \geqslant 0 \tag{6.7}$$

or

$$\beta \geqslant -\frac{1}{4j^2} \qquad \forall\ j \quad (j \neq 0). \tag{6.8}$$

Such unitary irreducible representations (6.5) and (6.6) are associated with explicit forms of the $sl_\beta(2)$-nonlinear generators expressed in terms of the undeformed $sl(2)$ ones. In fact, we formally claim that, according to (2.9) and (6.5), we have

$$\hat{J}_+ = J_+\left(1 + 2\beta\left(\mathcal{C} + J_3(J_3+1)\right)\right)^{1/2} = \mathcal{Q}(\mathcal{C}, J_+, J_3) \tag{6.9}$$

and

$$\hat{J}_- = \left(1 + 2\beta\left(\mathcal{C} + J_3(J_3+1)\right)\right)^{1/2} J_- = (\hat{J}_+)^+ \tag{6.10}$$

while the third one $\hat{J}_3$ is unchanged (see equation (2.10)). Here we point out that the corresponding ϕ-function (2.26) is not bijective and the corresponding Hopf structure cannot be written using our β-generators.

Let us now insist on an interesting property which, to our knowledge, seems not yet to have been exploited, i.e. on a possible shift of the diagonal generator spectrum expressed in terms of a (real scalar) parameter called hereafter γ. So, let us propose a modification of relation (6.6) in the following way:

$$\hat{J}_3|j,m\rangle = (m+\gamma)|j,m\rangle. \tag{6.11}$$

If it is evident that in the usual angular momentum theory such a shift has no physical meaning; it is non-trivial to show that, in a q-deformed one, nothing more happens when q is not a root of unity. Indeed, if we require the commutation relation

$$[J_+, J_-] = [2J_3] \tag{6.12}$$

with the bracket (4.6) and if we require

$$J_+|j,m\rangle = \sqrt{f(j,m)}\;|j,m+1\rangle \tag{6.13}$$

$$J_+|j,m\rangle = \sqrt{f(j,m-1)}\;|j,m-1\rangle \tag{6.14}$$

when

$$J_3|j,m\rangle = (m+\gamma)|j,m\rangle \tag{6.15}$$

it is possible to show that

$$f(j,m) = \frac{1}{(q-q^{-1})^2}\left(q^{-2j+2\gamma-1} + q^{2j-2\gamma+1} - q^{2m+2\gamma+1} - q^{-2m-2\gamma-1}\right). \tag{6.16}$$

Then, due to the fact that, from (6.13), we have

$$f(j,j) = 0 \tag{6.17}$$

from (6.16) we get

$$f(j,j) = [-2\gamma][2j+1] = 0 \tag{6.18}$$

asking for the annulation of the parameter γ. We thus conclude that the shift (6.15) does not allow us to characterize new representations of $\mathcal{U}_q(sl(2))$.

The study of the Higgs algebra in that direction is richer and non-zero values of γ can be exploited in order to select new unitary irreducible representations of this cubic $sl(2)$ algebra. In order to establish such a result, let us consider equation (6.11) within the Higgs context characterized by the commutation relations (6.2) and (6.3). The action of the ladder operators $\hat{J}_\pm$ on the basis leads to γ-dependent $f^\pm$-functions. In fact, in correspondence with equations (6.5), here we get

$$\hat{J}_+|j,m\rangle = \Big((j-m)(j+m+1+2\gamma)\left(1+2\beta\big(j(j+1)+m(m+1)\right. \\ \left.\left. + 2\gamma(j+m+1+\gamma)\big)\right)\Big)^{1/2}|j,m+1\rangle \tag{6.19}$$

and

$$\hat{J}_-|j,m\rangle = \Big((j-m+1)(j+m+2\gamma)\,(1+2\beta(j(j+1)+m(m-1) \\ + 2\gamma(j+m+\gamma)))\Big)^{1/2}|j,m-1\rangle. \tag{6.20}$$

By exploiting the property that

$$\hat{J}_-|j,-j\rangle = 0 \tag{6.21}$$

we get the constraint

$$2\gamma(2j+1)\left(1+4\beta(j(j+1)+\gamma^2)\right) = 0 \tag{6.22}$$

showing that, besides our preceding context ($\gamma = 0$), there are other possibilities related to non-zero γ values issued from the equation

$$\gamma^2 = \frac{1}{4\beta^2}\left(-\beta - 4\beta^2\ j(j+1)\right). \tag{6.23}$$

A simple discussion of its roots leads to the *two* families of new representations characterized respectively by

$$\gamma = \frac{1}{2\beta}\left(-\beta - 4\beta^2\ j(j+1)\right)^{1/2} \tag{6.24}$$

or

$$\gamma = -\frac{1}{2\beta}\left(-\beta - 4\beta^2\ j(j+1)\right)^{1/2} \tag{6.25}$$

both values being constrained by the deformation parameter β such that

$$-\frac{1}{4j(j+1)} < \beta \leqslant -\frac{1}{4j(j+1)+1}. \tag{6.26}$$

Let us insist on the fact that these representations are typical of the deformation characterizing the Higgs algebra: they do not exist when $\beta = 0$. Moreover, such a method suggests its application to other nonlinear $sl(2)$ algebras and here we want to look at its impact on an interesting quadratic one [4] in the following section.

Just as the simplest example, let us fix $j = \frac{1}{2}$ (corresponding to the fundamental representation in the conventional $sl(2)$ case). We evidently conclude that, if our β-parameter is constrained (according to (6.26)) by

$$-\tfrac{1}{3} < \beta \leqslant -\tfrac{1}{4} \tag{6.27}$$

we get *three* families of representations corresponding to

$$\gamma = \pm\frac{1}{2\beta}\left(-\beta - 3\beta^2\right)^{1/2} \qquad \text{and} \qquad \gamma = 0. \tag{6.28}$$

According to (6.8) when $\gamma = 0$, we have $\beta \geqslant -1$ and we point out that, if $\beta > \frac{1}{4}$ or if $-1 \leqslant \beta \leqslant -\frac{1}{3}$, we get only one family while, evidently, if $\beta < -1$, no representation is admissible.

As a last remark, let us notice that the modification effectively introduced in (6.11) through the γ-parameter does not affect our conclusions regarding the Hopf structure of the Higgs algebra.

7. The quadratic $sl(2)$ algebra

Another nonlinear $sl(2)$ algebra is the *quadratic* one [4] characterized by the following commutation relations depending on the (real scalar) parameter α:

$$[J_+^{(\alpha)}, J_-^{(\alpha)}] = 2J_3^{\alpha} + 4\alpha(J_3^{(\alpha)})^2 \tag{7.1}$$

$$[J_3^{(\alpha)}, J_\pm^{(\alpha)}] = \pm J_\pm^{(\alpha)}. \tag{7.2}$$

It has already been exploited [4] in connection with Yang–Mills-type gauge theories and with fundamental quantum mechanical problems [5, 6]. In particular, its representation theory has already been investigated [5] for the lowest eigenvalues of the Casimir operator.

Let us return to this representation theory when combined with the demand corresponding to (6.11) of the preceding section, i.e.

$$J_3^{(\alpha)}|j,m\rangle = (m+\gamma)|j,m\rangle. \tag{7.3}$$

Here the ladder operators $J_\pm^{(\alpha)}$ also act on the basis and determine α-dependent $f^\pm$-functions that can be calculated. They are given in the following relations:

$$J_+^{(\alpha)}|j,m\rangle = \Big((j-m)\big(j+m+1+2\gamma+\alpha\big(\tfrac{4}{3}j^2+\tfrac{4}{3}jm + \tfrac{4}{3}m^2+4\gamma j+4\gamma m+2j+2m+4\gamma^2+4\gamma+\tfrac{2}{3}\big)\big)\Big)^{1/2}|j,m+1\rangle \tag{7.4}$$

and

$$J_-^{(\alpha)}|j,m\rangle = \Big((j-m+1)\big(j+m+2\gamma+\alpha\big(\tfrac{4}{3}j^2+\tfrac{4}{3}jm\big) + \tfrac{4}{3}m^2+4\gamma j+4\gamma m+\tfrac{2}{3}j-\tfrac{2}{3}m+4\gamma^2\big)\big)\Big)^{1/2}|j,m-1\rangle. \tag{7.5}$$

Once again, the condition

$$J_-^{(\alpha)}\ |j,-j\rangle = 0 \tag{7.6}$$

leads to the constraint

$$\gamma = \frac{1}{4\alpha}\left(-1+\sqrt{1-\frac{16}{3}j(j+1)\,\alpha^2}\right) \tag{7.7}$$

when

$$\alpha \leqslant \frac{3}{2(4j+1)}. \tag{7.8}$$

Such unitary irreducible representations (7.3)–(7.8) are typical of the deformation and are associated with the following forms of generators explicitly given in terms of the undeformed $sl(2)$ ones:

$$J_3^{(\alpha)} = J_3 - \frac{1}{4\alpha} + \frac{1}{4\alpha}\sqrt{1-\frac{16}{3}\,\alpha^2\,\mathcal{C}} \tag{7.9}$$

$$J_+^{(\alpha)} = J_+\ \left(\tfrac{2}{3}\,\alpha\,(2J_3+1)+\sqrt{1-\tfrac{16}{3}\,\alpha^2\,\mathcal{C}}\right)^{1/2} \tag{7.10}$$

$$J_-^{(\alpha)} = \left(\tfrac{2}{3}\,\alpha\,(2J_3+1)+\sqrt{1-\tfrac{16}{3}\,\alpha^2\,\mathcal{C}}\right)^{1/2}\ J_-. \tag{7.11}$$

Through knowledge of the $sl(2)$-coproduct, counit and antipode given by (5.1)–(5.8), we can thus provide the quadratic algebra (7.1) and (7.2) with a Hopf structure by defining

$$\triangle(J_3^{(\alpha)}) = \triangle(J_3) - \frac{1}{4\alpha}\,(1\otimes 1) + \frac{1}{4\alpha}\sqrt{1\otimes 1-\frac{16}{3}\,\alpha^2\ \triangle(\mathcal{C})} \tag{7.12}$$

$$\triangle(J_+^{(\alpha)}) = \triangle(J_+)\left(\tfrac{2}{3}\,\alpha\,(2\triangle(J_3)+1\otimes 1)+\sqrt{1\otimes 1-\tfrac{16}{3}\,\alpha^2\,\triangle(\mathcal{C})}\right)^{1/2} \tag{7.13}$$

$$\triangle(J_-^{(\alpha)}) = \left(\tfrac{2}{3}\ \alpha\ (2 \triangle (J_3) + 1 \otimes 1) + \sqrt{1 \otimes 1 - \tfrac{16}{3}\alpha^2 \triangle (\mathcal{C})}\right)^{1/2} \triangle (J_-) \tag{7.14}$$

$$\varepsilon(J_3^{(\alpha)}) = \varepsilon(J_\pm^{(\alpha)}) = 0 \tag{7.15}$$

$$S(J_3^{(\alpha)}) = -J_3 - \frac{1}{4\alpha} + \frac{1}{4\alpha}\sqrt{1 - \frac{16}{3}\alpha^2\mathcal{C}} \tag{7.16}$$

$$S(J_+^{(\alpha)}) = -\left(\tfrac{2}{3}\alpha(-2J_3 + 1) + \sqrt{1 - \tfrac{16}{3}\alpha^2\mathcal{C}}\right)^{1/2} J_+ \tag{7.17}$$

$$S(J_-^{(\alpha)}) = -J_- \left(\tfrac{2}{3}\alpha(-2J_3 + 1) + \sqrt{1 - \tfrac{16}{3}\alpha^2\mathcal{C}}\right)^{1/2} \tag{7.18}$$

as was the case for the cubic algebra (6.2) and (6.3) but with the definitions (5.1)–(5.2). We note that the right-hand sides of (7.12)–(7.18) cannot be written using only the generators $J_\pm^{(\alpha)}$ and $J_3^{(\alpha)}$.

8. Conclusions and comments

We have developed the representation theory associated with *nonlinear* $sl(2)$ algebras characterized by the structure relations (2.7) and (2.8) containing, in particular, the linear $sl(2)$ algebra as well as its q-deformation $\mathcal{U}_q(sl(2))$. Moreover, we have more specifically visited the *cubic* $sl(2)$ algebra in order to get *all* its unitary irreducible representations and to show that it is endowed in our formalism with a Hopf structure, the corresponding results also being presented for the *quadratic* $sl(2)$ algebra. Such a study mainly takes advantage of the fact that we can express the generators of the nonlinear algebras in terms of the old (undeformed) $sl(2)$ ones and that the $sl(2)$ algebra is endowed with a well known Hopf structure. These properties allow us to extend our considerations for arbitrary N in the odd case (developed in section 2) and are also valid in principle for the even context after the study of the $N = 2$ case (developed in section 7).

From the representation point of view, our results generalize to arbitrary j's those obtained by Roček [5]. They also include others obtained by Zhedanov [14], Feng Pan [15] and Bonatsos *et al* [16].

From the point of view of Hopf structures associated with our developments, many connections with recent studies can be pointed out. An interesting property discussed in section 5 is that concerning the cocommutativity or non-cocommutativity of the already known coproducts. We have shown that, in some particular cases, the nonlinear algebra can be equipped with a consistent Hopf structure (i.e. the corresponding coproduct being expressed in terms of *deformed* generators). Moreover, let us mention that there is also a third possibility by exploiting our recent proposal for a new deformed structure $\mathcal{U}_q^\theta(sl(2))$ algebra using a real para-Grassmannian variable θ [17].

All these properties have to be carefully examined and we plan to come back to these in the future. Let us finally add that our results, in particular, confirm those recently obtained by Quesne and Vansteenkiste [18], showing that if we ask for a deformed coproduct in terms of deformed generators, only the already well known ones are possible. We have obtained new ones due to the fact that we have expressed the *deformed* generators (in each context) in terms of the undeformed ones.

Acknowledgments

We want to thank J F Cornwell for drawing our attention to [13]. This work has been elaborated with the financial support of a TOURNESOL grant (which is also acknowledged by all of us).

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